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Integral representations for the exceptional univariate Lommel functions

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Abstract
An elementary integral representation for the Lommel function $S_{-1,0}(z)$ is given and extended to other exceptional cases.

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1. Introduction

The Lommel function [1] generally denoted by $S_{\mu,\nu}(z)$ is a particular solution to the inhomogeneous Bessel equation

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1} \tag{1}$$

and occurs in a variety of studies in physics and engineering [2–5]. The standard reference formulas for this function (e.g. [6–8]) are restricted to the cases where $\mu \pm \nu$ is not a negative odd integer. To the author's knowledge the only detailed description of the exceptional case $S_{-1,0}(z)$ is by Watson [1], who gives the explicit expression

$$S_{-1,0}(z) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(n!)^2} \left\{ [\ln(z/2) - \psi(k+1)]^2 - \frac{1}{2}\psi'(k+1) + \frac{\pi^2}{4} \right\}. \tag{2}$$

Representations for this exceptional case, $S_{-1,0}(z)$, are less well documented [9][1] and the aim of this work is to provide a simple alternative derivation.

[1] This formula is incorrect as printed and should read

$$S_{\mu,\nu}(z) = z^{\mu+1} \int_0^{\infty} t\, e^{-zt}\, {}_2F_1\left(\frac{1-\mu+\nu}{2}, \frac{1-\mu-\nu}{2}; \frac{3}{2}; -t^2\right) dt.$$

2. Derivation

We begin with an integral identity, which appears to be new and of broad utility.

Theorem 1. *For* $|Re[\mu,\nu]| < 2$ *and* $a > 0$

$$\int_0^\infty K_\nu(a\sinh t)\frac{\sinh(\mu t)}{\sin(\mu\pi/2)}\,\mathrm{d}t = \int_0^\infty K_\mu(a\sinh t)\frac{\sinh(\nu t)}{\sin(\nu\pi/2)}\,\mathrm{d}t. \tag{3}$$

Proof. From [10] we have, under the conditions stated,

$$\int_0^\infty K_\nu(ax)\sin(xy)\,\mathrm{d}x = \frac{\pi}{4}a^{-\nu}\csc(\nu\pi/2)\frac{[\sqrt{a^2+y^2}+y]^\nu - [\sqrt{a^2+y^2}-y]^\nu}{\sqrt{a^2+y^2}} \tag{4}$$

and the similar formula with ν and a replaced by μ and b, respectively. Therefore, by the Parseval identity for the Fourier transform,

$$\frac{b^{-\mu}}{\sin(\mu\pi/2)}\int_0^\infty \frac{K_\nu(ax)}{\sqrt{b^2+x^2}}\{[\sqrt{b^2+x^2}+x]^\mu - [\sqrt{b^2+x^2}-x]^\mu\}\,\mathrm{d}x$$
$$= \frac{a^{-\nu}}{\sin(\nu\pi/2)}\int_0^\infty \frac{K_\mu(by)}{\sqrt{a^2+y^2}}\{[\sqrt{a^2+y^2}+y]^\nu - [\sqrt{a^2+y^2}-y]^\nu\}\,\mathrm{d}y. \tag{5}$$

Next, on the right-hand side of (5) scale $y \to au$, on the left-hand side scale $x \to bu$ and replace ab by a. Finally, the substitution $u = \sinh t$ on both sides produces (3). □

Writing $\mu = \mathrm{i}x$ we obtain

Corollary 1. *For* x *real,* $a > 0$*,*

$$\int_0^\infty K_{ix}(a\sinh t)\frac{\sinh(\nu t)}{\sin(\nu\pi/2)}\,\mathrm{d}t = \int_0^\infty K_\nu(a\sinh t)\frac{\sin(xt)}{\sinh(\pi x/2)}\,\mathrm{d}t. \tag{6}$$

Equation (6) is of interest with respect to the Kontorovich–Lebedev (K-L) transform as it can replace integration over the complex index to integration over a real argument of an elementary function.

Theorem 2. *For* $Re[z] > 0$

$$S_{-1,0}(z) = \int_0^\infty t\,\mathrm{e}^{-z\sinh t}\,\mathrm{d}t. \tag{7}$$

Proof. We note the known K-L transform [11]

$$\int_0^\infty \cos(bx)K_{ix}(y)\,\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-y\cosh b} \tag{8}$$

and the elementary result

$$\int_0^\infty \frac{\cos(bx)\sin(tx)}{\sinh(\pi x/2)}\,\mathrm{d}x = \frac{\sinh(2t)}{\cosh(2b)+\cosh(2t)}. \tag{9}$$

Then multiplying both sides of (6) by $\cos(bx)$ and integrating over x, where the change of the integration order is allowed by absolute convergence, the result can be manipulated into the form

$$\frac{\pi}{2}\int_0^\infty \mathrm{e}^{-a\cosh(b)\sinh(t)}\frac{\sinh(\nu t)}{\sin(\pi\nu/2)}\,\mathrm{d}t = a^{-1/2}\int_0^\infty \frac{x^{1/2}}{\cosh^2 b + x^2}\sqrt{ax}\,K_\nu(ax)\,\mathrm{d}x. \tag{10}$$

Now we take the limit $\nu \to 0$ and note that the right-hand integral in (10) is a tabulated K-transform [12] to obtain

$$\int_0^\infty t\,\mathrm{e}^{-a\cosh(b)\sinh t}\,\mathrm{d}t = S_{-1,0}(a\cosh b) \tag{11}$$

which is just (7) with $a\cosh b = z$. □

3. Discussion

The best known elementary integral representations for the functions $S_{\mu,\nu}(z)$ appear to be due to Szymanski [13], who found, following Hankel's work for the ordinary Bessel functions,

$$S_{\mu,\nu}(z) = z^{\mu+1} \int_{(0,+1,1+\mathrm{i}\infty)} \mathrm{e}^{\mathrm{i}zt} W_{\mu,\nu}(t)\, \mathrm{d}t \tag{12}$$

$$W_{\mu,\nu}(t) = (-1)^{(\nu-\mu-2)/2} \frac{\sqrt{\pi}\Gamma[(\mu+\nu+1)/2]}{2^{\nu-\mu}\Gamma(\nu+1/2)\Gamma[(\nu-\mu+1)/2]} \frac{\mathrm{d}^{\nu-\mu-1}}{\mathrm{d}t^{\nu-\mu-1}}(1-t^2)^{\nu-1/2}, \tag{13}$$

which is invalid for the exceptional case because of the Γ-function in the numerator of (13).

By a simple change of variable and integration by parts, (7) can be expressed as the Laplace transforms

$$\int_0^\infty \frac{\sinh^{-1} x}{\sqrt{x^2+1}} \mathrm{e}^{-zx}\, \mathrm{d}x = S_{-1,0}(z) \tag{14}$$

$$\int_0^\infty [\sinh^{-1} x]^2 \mathrm{e}^{-zx}\, \mathrm{d}x = \frac{2}{z} S_{-1,0}(z). \tag{15}$$

The availability of recursion relations such as

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\nu}{z}\right] S_{\mu,\nu}(z) = (\mu - \nu - 1) S_{\mu-1,\nu+1}(z) \tag{16}$$

provides similar representations for all other exceptional cases; for example,

$$S_{-2,1}(z) = \frac{1}{2}\int_0^\infty t \sinh t\, \mathrm{e}^{-z\sinh t}\, \mathrm{d}t = \frac{1}{2}\int_0^\infty \frac{x \sinh^{-1} x}{\sqrt{x^2+1}} \mathrm{e}^{-zx} \mathrm{d}x. \tag{17}$$

Finally, the existence of such a simple representation as (7) can streamline the evaluation of integrals containing $S_{-1,0}(z)$, since for arbitrary f

$$\int_0^\infty f(x) S_{-1,0}(x)\, \mathrm{d}x = \int_0^\infty \frac{\sinh^{-1} x}{\sqrt{x^2+1}} F(x)\, \mathrm{d}t \tag{18}$$

where F denotes the Laplace transform of f and the interchange of the integration order is allowed. An interesting example is

$$\phi(\nu) = \int_0^\infty x^\nu J_\nu(x) S_{-1,0}(x)\, \mathrm{d}x = 2^\nu \frac{\Gamma(\nu+1/2)}{\sqrt{\pi}} \int_0^\infty \frac{\sinh^{-1} x}{(1+x^2)^{\nu+1}}\, \mathrm{d}x. \tag{19}$$

Thus, $\phi(0) = 2\mathbf{G}$ and $\phi(1/2) = \sqrt{2/\pi}$. This extends the calculation of Aslam Chaudhry [14], which was, in part, the motivation for this work.

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